JOURNAL OF APPROXIMATION THEORY 14, 230–243 (1975)

Sard's Method and the Theory of Spline Systems

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Communicated by Arthur Sard

0. INTRODUCTION

In his paper concerning the theory of optimal approximation [12], A. Sard has developed a method for the construction of spline approximations in an abstract setting. Sard's method provides a unified approach to the minimal property in the ground space and to the dual minimal property of the classical polynomial spline functions (cf. Schoenberg [13], [14]). On the other hand the authors have introduced in [6], [7] the concept of spline system to obtain the principal "intrinsic" properties of the classical spline theory from a purely functional analytic point of view. *In the present paper one of our basic aims is to study some connections between the method of Sard which will be described by the notion of Sard system* (see Section 1) *and the concept of spline system.*

Section 1 is devoted to a concise description of Sard's method and to a formulation of a general Dirichlet principle with generalized boundary conditions. Moreover, it contains the construction of a spline system associated with a given Sard system and establishes dual minimal properties for spline approximations. Then, in Section 2, we present as an application of the results derived in Section 1 the Sard systems of Hermite cubic splines, of periodic cubic splines, and of the Dirichlet problem in $\mathbf{R}^n$, and describe the spline systems associated with them. Section 3 studies some convergence properties of spline approximations which are constructed by means of spline systems. Finally, Section 4 collects some additional remarks.

For a general survey of the theory of spline functions we refer the reader to the monographs by Ahlberg–Nilson–Walsh [2] and Laurent [10].

1. THE METHOD OF SARD AND THE ASSOCIATED SPLINE SYSTEM

Let us begin with the definition of Sard systems. Suppose that $(X; \|\cdot\|)$ denotes a Banach space and that $(Y; (\cdot \mid \cdot)_Y)$, $(Z_j\,; (\cdot \mid \cdot)_{Z_j})$ $(1 \leqslant j \leqslant m)$ form a family of prehilbert spaces. We emphasize that, throughout this paper, all vector spaces are considered to be modelled over the field $\mathbf{R}$ of real numbers. Let

$$U \in \mathscr{L}(X, Y);$$
$$F_j \in \mathscr{L}(X, Z_j) \qquad (1 \leqslant j \leqslant m)$$

be *norm continuous* $\mathbf{R}$-linear mappings. The tuplet

$$(X; Y; Z_1, ..., Z_m; U; F_1, ..., F_m) \tag{1}$$

is called a (real) *Sard system* iff there exists a constant $B > 0$ such that the following inequality of Friedrichs type

$$\| x \|^2 \leqslant B\left(\| U(x)\|_Y^2 + \sum_{1\leqslant j\leqslant m} \| F_j(x)\|_{Z_j}^2\right) \tag{2}$$

holds for all elements $x \in X$.

Define a scalar product on X according to the Golomb–Weinberger procedure [9] as follows:

$$(\cdot \mid \cdot)_X = (U(\cdot) \mid U(\cdot))_Y + \sum_{1\leqslant j\leqslant m} (F_j(\cdot) \mid F_j(\cdot))_{Z_j}. \tag{3}$$

Then $(X; (\cdot \mid \cdot)_X)$ becomes a Hilbert space. Clearly X under the graph norm $\| \cdot \|_X$ which is canonically induced by the scalar product $(\cdot \mid \cdot)_X$ is *toplinear isomorphic* to the initially given space $(X; \| \cdot \|)$. It follows that there exists a unique *orthogonal projector* P_m of the Hilbert space X satisfying the condition

$$\operatorname{Im}(P_m) = \left(\bigcap_{1\leqslant j\leqslant m} \operatorname{Ker}(F_j)\right)^{\perp}.$$

Here $\operatorname{Im}(\cdot)$ (resp. $\operatorname{Ker}(\cdot)$) denotes as usual the image (resp. the kernel) of the linear mapping under consideration and $(\cdot)^{\perp}$ stands for the orthogonal supplement with respect to the scalar product $(\cdot \mid \cdot)_X$. The linear mapping $P_m \in \mathscr{L}(X) = \mathscr{L}(X, X)$ is called the *spline projector* of the Sard system (1) and for any $x_0 \in X$ the element $P_m(x_0) \in X$ is referred to as the *spline approximation* of x_0. By the projection theorem, the latter satisfies the following *general Dirichlet principle*:

THEOREM 1 (Sard [12]). *Let any element $x_0 \in X$ be given. The spline approximation $P_m(x_0)$ of x_0 is the unique element in the set of all $x \in X$ subject to the "generalized boundary conditions"*

$$F_j(x) = F_j(x_0) \qquad \text{for} \quad 1 \leqslant j \leqslant m,$$

which minimizes the Dirichlet functional

$$x \rightsquigarrow \| U(x)\|_Y.$$

Using the terminology of Sard [12], P_m represents the optimal approximation of the identity automorphism id_X. It should be observed, however, that the hypotheses of Sard concerning the completeness of the ranges $\mathrm{Im}(F_j)$ in Z_j $(1 \leqslant j \leqslant m)$ are *not* necessary for the validity of the theorem supra.

The following theorem establishes a connection between Sard's method as described by the notion of Sard system and the concept of *spline system.* The definition of the latter one will not be repeated here. It can be found in the papers [6], [7] cited in Section 0.

THEOREM 2. *Let P_m be the spline projector of the Sard system*

$$(X; Y; Z_1, \ldots, Z_m; U; F_1, \ldots, F_m).$$

If $\mathrm{Im}(U)$ *is a complete vector subspace of the prehilbert space* Y, *then*

$$(X, P_m, U, \mathrm{Im}(U)) \tag{4}$$

represents a (real) spline system.

Proof. We have to check the defining properties (I)–(IV) of spline systems.

(I) It follows from the definition that $P_m \in \mathscr{L}(X)$ is an idempotent endomorphism of X.

(II) Since $\mathrm{Im}(U)$ is assumed to be complete, the open-mapping theorem shows that $U \in \mathscr{L}(X, \mathrm{Im}(U))$ is an epimorphism, i.e., a continuous open surjective linear mapping.

(III) For any $x_0 \in X$ satisfying $F_j(x_0) = 0$ for $1 \leqslant j \leqslant m$ and any point $x \in \mathrm{Ker}(U)$, we have obviously $(x_0 \mid x)_X = 0$. Consequently the inclusion

$$\mathrm{Ker}(U) \subseteq \Big(\bigcap_{1 \leqslant j \leqslant m} \mathrm{Ker}(F_j) \Big)^{\perp}$$

obtains. Thus, the kernel condition

$$\mathrm{Ker}(U) \subseteq \mathrm{Im}(P_m)$$

is established.

(IV) Let $P_m' = \mathrm{id}_X - P_m$ be the supplementary orthogonal projector corresponding to P_m. Obviously (3) yields the identity

$$(U \circ P_m(x_1) \mid U \circ P_m'(x_2))_Y = 0$$

for any pair $(x_1, x_2) \in X \times X$. Thus the orthogonality relation

$$\mathrm{Im}(U \circ P_m) \perp \mathrm{Im}(U \circ P_m')$$

holds and the proof is complete.

In the following we shall refer to (4) as the *spline system associated with the Sard system* (1). As is well known from the Banach epimorphism theorem, it is sufficient for the validity of Theorem 2 that $\mathrm{Im}(U)$ is assumed to be nonmeager (i.e., of the second Baire category) in itself. For applications of Theorem 2, however, the next theorem is of greater importance.

THEOREM 3. *Let the Sard system* (1) *be given. If for an integer m_0, where $1 \leqslant m_0 \leqslant m$, the tuplet*

$$(X; Y; Z_1, ..., Z_{m_0}; U; F_1, ..., F_{m_0}) \tag{5}$$

is a Sard system satisfying the condition

$$\mathrm{Ker}(U) = \Big(\bigcap_{1 \leqslant j \leqslant m_0} \mathrm{Ker}(F_j) \Big)^{\perp} \tag{6}$$

then $\mathrm{Im}(U)$ *is a complete vector subspace of Y.*

Proof. Consider the Sard system (5) and let $(y_n)_{n \geqslant 1}$ be a Cauchy sequence in $\mathrm{Im}(U)$. Choose a sequence $(x_n)_{n \geqslant 1}$ in $\mathrm{Ker}(U)^{\perp}$ such that $U(x_n) = y_n$ for any index $n \geqslant 1$. Combining (2) and (6) we see that $(x_n)_{n \geqslant 1}$ is a Cauchy sequence in the space X. Let $x_0 \in X$ be its limit. Then $(y_n)_{n \geqslant 1}$ converges in $\mathrm{Im}(U)$ towards the point $y_0 = U(x_0)$.

For another sufficient condition which guarantees the completeness of $\mathrm{Im}(U)$ and which is based on an inequality of the Poincaré type, see [5].

Suppose that $\mathrm{Im}(U)$ is complete. As we have stated in [6], [7], the minimal properties of the spline system $(X, P_m, U, \mathrm{Im}(U))$ in the ground space take the following form: For any $x_0 \in X$ the inequalities

$$\begin{aligned} \| U(x_0) - U \circ P_m(x_0) \|_Y &\leqslant \| U(x_0) - U \circ P_m(x) \|_Y, \\ \| U(x_0) - U \circ P_m'(x_0) \|_Y &\leqslant \| U(x_0) - U \circ P_m'(x) \|_Y \end{aligned} \tag{7}$$

hold for all points $x \in X$. Since P_m' is the orthogonal projection in the Hilbert space X on its vector subspace

$$\bigcap_{1 \leqslant j \leqslant m} \mathrm{Ker}(F_j),$$

the second one of the inequalities (7) implies the statement of Theorem 1. On the other hand, the *dual* minimal property of spline systems, i.e., the generalized Schoenberg theorem yields a dual minimal property for the spline projector of Sard systems. For its explicit formulation we have to introduce the *transposed* linear mappings between the associated *strong topological duals*

$$\begin{aligned} {}^tU &\in \mathscr{L}(Y', X'); \\ {}^tP_m &\in \mathscr{L}(X'); \\ {}^tP_m' &\in \mathscr{L}(X'). \end{aligned}$$

Moreover, let us introduce the (continuous) inverse linear mapping

$$W = ({}^tU)^{-1} \in \mathscr{L}(\operatorname{Im}({}^tU), Y').$$

We shall suppose without any loss of generality that $\operatorname{Im}(U) = Y$, i.e., that $U \in \mathscr{L}(X, Y)$ represents an epimorphism. Then, according to [6], for any $x_0' \in X'$ the inequality

$$\| W \circ {}^tP_m'(x_0')\|_{Y'} \leqslant \| W(x_0' - {}^tP_m(x'))\|_{Y'}$$

holds for all points $x' \in x_0' + \operatorname{Im}({}^tU)$. If we observe the identity

$$\operatorname{Im}({}^tU) = \operatorname{Ker}(U)^0,$$

where $(\cdot)^0$ stands for the polar with respect to the canonical bilinear form $(x, x') \rightsquigarrow \langle x, x'\rangle$ of the real duality (X, X'), we obtain the following result:

THEOREM 4 (dual minimal property). *Suppose that* $\operatorname{Im}(U) = Y$ *and let* Y *be complete (i.e., a Hilbert space). Fix any continuous linear form* $x_0' \in X'$. *Then* ${}^tP_m(x_0')$ *is the unique linear form in the set of all* $x' \in X'$ *subject to the exactness condition*

$$\langle z, x'\rangle = \langle z, x_0'\rangle \qquad \textit{for all} \quad z \in \operatorname{Ker}(U),$$

which minimizes the functional

$$x' \rightsquigarrow \| W(x_0' - {}^tP_m(x'))\|_{Y'} .$$

The foregoing theorem is a general functional analytic form of Schoenberg's approximation theorem and represents an additional aspect of Sard's theory. In the next section we shall point out some concrete applications of these results. For this purpose, the following notion reveals to be useful:

The Sard system (1) is said to satisfy a *Poincaré condition* provided that $\operatorname{Im}(U) = Y$ and that there exist a prehilbert space $(Z; (\cdot \mid \cdot)_Z)$ and a linear mapping $F \in \mathscr{L}(X, Z)$ which have the following properties:

(i) *$(X; Y; Z; U; F)$ is a Sard system*;

(ii) *The relations*

$$\bigcap_{1 \leqslant j \leqslant m} \operatorname{Ker}(F_j) \subseteq \operatorname{Ker}(F); \tag{8}$$

$$\operatorname{Ker}(U) = \operatorname{Ker}(F)^{\perp} \tag{9}$$

are valid where $\perp$ *denotes orthogonality with respect to the scalar product*

$$((\cdot \mid \cdot))_X = (U(\cdot) \mid U(\cdot))_Y + (F(\cdot) \mid F(\cdot))_Z$$

on the vector space X.

Observe that the identity (9) together with Theorem 3 implies that Y is a Hilbert space. Thus, by Theorem 2, the spline system

$$(X, P_m, U, Y)$$

associated with (1) can be formed.

EXAMPLE. Let Ω be a relatively compact domain in the real Euclidean n-space $\mathbf{R}^n$ whose *boundary* $\partial\Omega$ *is* $\mathscr{C}$ *embedded in* $\mathbf{R}^n$. If j denotes the canonical injection

$$\mathrm{W}^{1,2}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$$

of the real Sobolev space $\mathrm{W}^{1,2}(\Omega)$ into the standard Lebesgue space $\mathrm{L}^2(\Omega) = \mathrm{L}^2(\Omega; \lambda^n)$ then

$$(\mathrm{W}^{1,2}(\Omega); \mathrm{Im}(\nabla); \mathrm{L}^2(\Omega); \nabla; j)$$

is a Sard system. If we combine it with

$$(\mathrm{W}^{1,2}(\Omega); \mathrm{Im}(\nabla); \mathbf{R}; \nabla; J)$$

where

$$J\colon \mathrm{W}^{1,2}(\Omega) \ni f \rightsquigarrow \int_\Omega f(x)\, \mathrm{d}\lambda^n(x),$$

the *classical Poincaré inequality* (see, for instance, Nečas [11, Chap. 1]) shows that it satisfies a Poincaré condition. See also Section 2, Example (iii).

THEOREM 5. *Let the Sard system* (1) *satisfy a Poincaré condition as described above. Then, for any* $x_0 \in X$, *the functional*

$$x \rightsquigarrow \| U(x_0 - P_m(x))\|_Y$$

where $x \in X$ *runs through the set of all vectors which satisfy the condition*

$$F(x) = F(x_0)$$

takes its minimum at $x = P_m(x_0)$.

For the proof of Theorem 5 we have to establish two lemmas.

LEMMA 1. *The continuous projector* P_m *in the Hilbert space* $(X; ((\cdot \mid \cdot))_X)$ *is selfadjoint, hence orthogonal.*

Proof. Part (IV) of the proof belonging to Theorem 2 shows that the orthogonality

$$(U \circ P_m(x_1) \mid U \circ P_m'(x_2))_Y = 0$$

holds for all pairs $(x_1, x_2) \in X \times X$. Since $P_m'(x_2) \in \bigcap_{1 \leqslant j \leqslant m} \operatorname{Ker}(F_j)$, the inclusion (8) yields

$$(F \circ P_m(x_1) \mid F \circ P_m'(x_2))_Z = 0.$$

Thus we have

$$((P_m(x_1) \mid P_m'(x_2)))_X = 0,$$

whence

$$((P_m(x_1) \mid x_2))_X = ((x_1 \mid P_m(x_2)))_X$$

for any pair $(x_1, x_2) \in X \times X$. Consequently $P_m = P_m^*$.

LEMMA 2 (U. Tippenhauer). *For the continuous linear mapping U of the Hilbert space $(X; ((\cdot \mid \cdot))_X)$ onto the Hilbert space $(Y; (\cdot \mid \cdot)_Y)$ and its adjoint U^*, the identity*

$$(U^*)^{-1}(x) = U(x)$$

holds for all points $x \in \operatorname{Ker}(F)$.

Proof. Let $\tilde{x} \in X$ be an arbitrary point. For any $x \in \operatorname{Ker}(F)$ we have $((x \mid \tilde{x}))_X = (U(x) \mid U(\tilde{x}))_Y$ and therefore

$$\begin{aligned} ((U^*)^{-1}(x) \mid U(\tilde{x}))_Y &= ((U^* \circ (U^*)^{-1}(x) \mid \tilde{x}))_X \\ &= ((x \mid \tilde{x}))_X \\ &= (U(x) \mid U(\tilde{x}))_Y . \end{aligned}$$

Since U is surjective, the conclusion follows.

To complete the proof of Theorem 5, we introduce the toplinear isomorphism

$$j_X \colon X \ni x \rightsquigarrow (X \ni \tilde{x} \rightsquigarrow ((\tilde{x} \mid x))_X) \in X'$$

of the Hilbert space $(X; ((\cdot \mid \cdot))_X)$ onto the Banach space X' and the canonical linear isometry

$$j_Y \colon Y \ni y \rightsquigarrow (Y \ni \tilde{y} \rightsquigarrow (\tilde{y} \mid y)_Y) \in Y'$$

of the Hilbert space $(Y; (\cdot \mid \cdot)_Y)$ onto its strong topological dual Y'. Then the identities

$$\begin{aligned} P_m^* &= (j_X)^{-1} \circ {}^tP_m \circ j_X , \\ U^* &= (j_X)^{-1} \circ {}^tU \circ j_Y \end{aligned}$$

hold. If we observe that

$$j_X(\operatorname{Ker}(F)) = \operatorname{Ker}(U)^0,$$

Theorem 4 together with Lemmas 1 and 2 establishes the result.

2. Three Examples of Spline Systems

(i) Let $I = [0, 1]$ be the compact unit interval of **R** with induced Lebesgue measure λ and $\mathscr{H}^2(I)$ the vector space consisting of the germs with respect to the neighborhood filter of I of all real-valued functions f in **R** for which the first derivative $\mathrm{D}f$ is absolutely continuous on I and $\mathrm{D}^2 f$ belongs to the Hilbert space $\mathrm{L}^2(I; \lambda) = \mathrm{L}^2(I)$. For any point $s \in I$ let ϵ_s denote the Dirac measure located at s and ϵ_s' its derivative in the sense of Schwartz distribution theory. Suppose that the space $\mathscr{H}^2(I)$ is equipped with the scalar product

$$(f, g) \rightsquigarrow \langle f, \epsilon_0 \rangle \cdot \langle g, \epsilon_0 \rangle + \langle f, \epsilon_0' \rangle \cdot \langle g, \epsilon_0' \rangle + \int_I \mathrm{D}^2 f(s) \cdot \mathrm{D}^2 g(s) \, d\lambda(s)$$

and the canonically induced norm $\|\cdot\|$. Observe that $\mathscr{H}^2(I)$ is toplinear isomorphic to the Sobolev space $\mathrm{W}^{2,2}(\mathring{I})$. Furthermore, let

$$\omega_0 = (s_\nu)_{0 \leqslant \nu \leqslant N} \qquad (N \geqslant 1)$$

be a given subdivision of I with the mesh points

$$0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1. \tag{10}$$

Then the tuplet

$$(\mathscr{H}^2(I); \mathrm{L}^2(I); \mathbf{R}, \ldots, \mathbf{R}; \mathrm{D}^2; \epsilon_{s_0}, \epsilon'_{s_0}, \epsilon_{s_1}, \ldots, \epsilon_{s_{N-1}}, \epsilon_{s_N}, \epsilon'_{s_N}) \tag{11}$$

represents a Sard system. Since the continuous linear mapping D^2: $\mathscr{H}^2(I) \to \mathrm{L}^2(I)$ is surjective, Theorem 2 is applicable. If SP_{ω_0} denotes the spline projector of (11), it shows that

$$(\mathscr{H}^2(I), \mathrm{SP}_{\omega_0}, \mathrm{D}^2, \mathrm{L}^2(I))$$

is a spline system. It follows from Theorem 1 combined with Holladay's theorem (see Ahlberg–Nilson–Walsh [2, Chap. III]) that the projector SP_{ω_0} is the *cubic spline interpolator* of Hermite type associated with the partition ω_0. In the present case, an application of Theorem 4 yields the *classical Schoenberg approximation theorem* for cubic splines as we have shown in [7] in the more general setting of L_m—splines. In this connection also see Atteia [4] and [5].

(ii) Let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ be the one-dimensional torus and μ the Haar measure on **T** normalized by $\mu(\mathbf{T}) = 1$. Construct the Hilbert space $\mathrm{L}^2(\mathbf{T}; \mu) = \mathrm{L}^2(\mathbf{T})$ and for the Lie group **T** the vector space $\mathscr{H}^2(\mathbf{T})$ of all real-valued functions $f \in \mathscr{C}^1(\mathbf{T})$ for which $f^{\#}: s \rightsquigarrow f(e^{2\pi i s})$ is absolutely continuous on I and $D^2 f \in L^2(\mathbf{T})$. Equip $\mathscr{H}^2(\mathbf{T})$ with the norm $\|\cdot\|$ derived from the following scalar product:

$$(f, g) \rightsquigarrow \langle f, \epsilon_1\rangle \cdot \langle g, \epsilon_1\rangle + \int_{\mathbf{T}} D^2 f(t) \cdot D^2 g(t)\, d\mu(t)$$

$$= f(1) \cdot g(1) + \int_I D^2 f^{\#}(s) \cdot D^2 g^{\#}(s)\, d\lambda(s).$$

See, for instance, Ehlich [8]. Moreover, denote by

$$\omega = (t_\nu)_{0 \leqslant \nu \leqslant N-1} \qquad (N \geqslant 1)$$

a partition of $\mathbf{T}$ which is not necessarily equidistant and which has the nodes

$$t_\nu = e^{2\pi i s_\nu} \qquad (0 \leqslant \nu \leqslant N-1).$$

The points $(s_\nu)_{0 \leqslant \nu \leqslant N-1}$ in I are assumed to be ordered as in (10). Then

$$(\mathscr{H}^2(\mathbf{T}); L^2(\mathbf{T}); \mathbf{R}, \ldots, \mathbf{R}; D^2; \epsilon_{t_0}, \ldots, \epsilon_{t_{N-1}})$$

represents a Sard system. Let SPP_ω be its spline projector and introduce the closed vector subspace

$$L_0^2(\mathbf{T}) = \{f \in L^2(\mathbf{T}) \mid \hat{f}(0) = 0\}$$

of $L^2(\mathbf{T})$, where

$$L^2(\mathbf{T}) \ni f \rightsquigarrow \hat{f} \in L^2(\mathbf{Z})$$

denotes the Fourier transformation. It follows that

$$(\mathscr{H}^2(\mathbf{T}), \mathrm{SPP}_\omega, D^2, L_0^2(\mathbf{T}))$$

is the spline system of *periodic cubic splines*. In particular, SPP_ω is the *periodic cubic spline interpolator*. An application of Theorems 1 and 4 yields its minimal properties as proved in Chapters III and V of Ahlberg–Nilson–Walsh [2]. We omit the details.

(iii) Let Ω be a relatively compact domain in the n-dimensional Euclidean space $(\mathbf{R}^n; |\cdot|)$ with *Lipschitz boundary* $\partial\Omega$. Let λ^n be the Lebesgue measure induced in Ω and σ the surface measure on $\partial\Omega$. Define the Sobolev space $W^{1,2}(\Omega)$ with its usual norm $f \rightsquigarrow \|f\|_{W^{1,2}(\Omega)}$, where

$$\|f\|^2_{W^{1,2}(\Omega)} = \|f\|^2_{L^2(\Omega)} + \sum_{1 \leqslant j \leqslant n} \left\| \frac{\partial}{\partial x_j} f \right\|^2_{L^2(\Omega)}.$$

Then the gradient

$$\nabla : W^{1,2}(\Omega) \to L^2(\Omega)^n$$

and the restriction mapping "*au sens des traces*" (Nečas [11, Chap. 1])

$$\mathrm{rest}_{\partial\Omega}\colon \mathrm{W}^{1,2}(\Omega) \to \mathrm{L}^2(\partial\Omega)$$

are continuous linear mappings. By virtue of the *classical Friedrichs inequality* (Nečas [11, Chap. 1]) there exists a constant $B > 0$ such that the estimate

$$\|f\|^2_{W^{1,2}(\Omega)} \leqslant B\left(\int_\Omega |\nabla f(x)|^2 \, d\lambda^n(x) + \int_{\partial\Omega} |f(x)|^2 \, d\sigma(x)\right)$$

holds for all functions $f \in \mathrm{W}^{1,2}(\Omega)$. Consequently we see that

$$(\mathrm{W}^{1,2}(\Omega);\ \mathrm{L}^2(\Omega)^n;\ \mathrm{L}^2(\partial\Omega);\ \nabla;\ \mathrm{rest}_{\partial\Omega}) \tag{12}$$

is a Sard system; cf. also [5] and Atteia [3]. If H_Ω denotes the corresponding spline projector, then Theorem 1 shows that for any function $f_0 \in \mathrm{W}^{1,2}(\Omega)$ its projection $H_\Omega(f_0)$ minimizes the "*energy integral*"

$$\int_\Omega |\nabla f(x)|^2 \, d\lambda^n(x)$$

when $f \in \mathrm{W}^{1,2}(\Omega)$ runs through the set of all functions which assume the same boundary values than f_0. Thus we obtain by the *classical Dirichlet principle* (Sobolev [15, Chap. II]) the following result:

THEOREM 6. *Let the function $f_0 \in \mathrm{W}^{1,2}(\Omega)$ be given. Then $H_\Omega(f_0) \in \mathrm{W}^{1,2}(\Omega)$ solves the Dirichlet problem*

$$\Delta H_\Omega(f_0) = 0;$$
$$\mathrm{rest}_{\partial\Omega}\, H_\Omega(f_0) = \mathrm{rest}_{\partial\Omega} f_0$$

for the pair $(\Omega, \mathrm{rest}_{\partial\Omega} f_0)$ in the sense of traces.

Next, define the continuous linear form

$$F\colon \mathrm{W}^{1,2}(\Omega) \ni f \rightsquigarrow \int_{\partial\Omega} f(x) \, d\sigma(x)$$

on the Sobolev space $\mathrm{W}^{1,2}(\Omega)$. If $C > 0$ denotes an appropriate constant, the following well known estimate (Sobolev [15, Chap. II])

$$\|f\|^2_{W^{1,2}(\Omega)} \leqslant C\left(\int_\Omega |\nabla f(x)|^2 + |F(f)|^2\right)$$

holds for all functions $f \in \mathrm{W}^{1,2}(\Omega)$. Thus

$$(\mathrm{W}^{1,2}(\Omega);\ \mathrm{Im}(\nabla);\ \mathbf{R};\ \nabla;\ F) \tag{13}$$

defines a Sard system. The system (12) together with (13) satisfies the hypotheses of Theorem 5. Hence this theorem makes the following *variational principle* apparent:

THEOREM 7. *Suppose that* $f_0 \in W^{1,2}(\Omega)$ *is given. Then the minimum of*

$$f \rightsquigarrow \| \nabla(f_0 - H_\Omega(f))\|_{L^2(\Omega)^n},$$

where $f \in W^{1,2}(\Omega)$ *runs through the set of all functions which satisfy the condition*

$$\int_{\partial\Omega} f(x)\, d\sigma(x) = \int_{\partial\Omega} f_0(x)\, d\sigma(x)$$

is assumed when $f = H_\Omega(f_0)$.

3. Convergence Properties of Spline Approximations

In addition to the minimal properties there are several well-known convergence properties of the classical spline functions. See Ahlberg–Nilson–Walsh [1], [2]. In the present section we shall deal with convergence properties of spline approximations constructed by means of spline systems via the method of Sard.

Let a *sequence*

$$((X; Y; Z_1, \ldots, Z_m; U; F_1, \ldots, F_m))_{m \geqslant 1}$$

of Sard systems be given which has $(P_m)_{m\geqslant 1}$ as its *sequence of spline projectors*. We shall suppose that U is surjective and that $Y = \mathrm{Im}(U)$ is a Hilbert space. Let X be endowed with the scalar product $(\cdot \mid \cdot)_X$ defined as in (3) by the Sard system $(X; Y; Z_1; U; F_1)$ and denote by P the orthogonal projector of the Hilbert space $(X; (\cdot \mid \cdot)_X)$ onto the closed vector subspace

$$\mathrm{Im}(P) = \left(\bigcap_{j\geqslant 1} \mathrm{Ker}(F_j)\right)^{\perp}.$$

THEOREM 8. *The tuplet*

$$(X, P, U, Y), \tag{14}$$

where P is defined as indicated above represents a spline system. For any $x_0 \in X$ *the convergence*

$$\lim_{m\to\infty} \| P_m(x_0) - P(x_0)\|_X = 0 \tag{15}$$

holds for the sequence $(P_m(x_0))_{m\geqslant 1}$ *of spline approximations of* x_0.

Proof. Let the vector space $\mathscr{L}(X)$ of continuous endomorphisms of X be equipped with the topology of pointwise convergence. Since, by Lemma 1 (which is indeed applicable), $(P_m)_{m\geqslant 1}$ is an increasing sequence of orthogonal projectors in $(X; (\cdot \mid \cdot)_X)$, it converges in $\mathscr{L}(X)$ towards an orthogonal projector Q in $(X; (\cdot \mid \cdot)_X)$ which has

$$\operatorname{Ker}(Q) = \bigcap_{m\geqslant 1} \operatorname{Ker}(P_m)$$

as its kernel. In view of the fact that the identity

$$\operatorname{Ker}(P_m) = \bigcap_{1\leqslant j\leqslant m} \operatorname{Ker}(F_j)$$

holds for any $m \geqslant 1$, we obtain

$$\operatorname{Ker}(Q) = \bigcap_{j\geqslant 1} \operatorname{Ker}(F_j).$$

Thus $P = Q$ and the pointwise convergence (15) follows.

Concerning the fact that (14) is a spline system, it is obviously sufficient to prove that $\operatorname{Ker}(U) \subseteq \operatorname{Im}(P)$ and that the orthogonality relation $\operatorname{Im}(U \circ P) \perp \operatorname{Im}(U \circ P')$ holds in the Hilbert space Y. By the general hypotheses we have made and by Theorem 2 we see that $((X, P_m, U, Y))_{m\geqslant 1}$ is a sequence of spline systems. Thus

$$\operatorname{Ker}(U) \subseteq \operatorname{Im}(P_m)$$

for any $m \geqslant 1$ and therefore

$$\operatorname{Ker}(U) \subseteq \operatorname{Im}(P).$$

Finally we have

$$(U \circ P_m(x_1) \mid U \circ P_m{}'(x_2))_Y = 0$$

for any pair $(x_1, x_2) \in X \times X$ and any $m \geqslant 1$. Thus the statement becomes evident.

EXAMPLES. We shall switch back to the Examples (i) and (ii) of Section 2 which are dealing with cubic splines. In the interval I let $(\omega_n)_{n\geqslant 1}$ be a sequence of partitions of type (10) such that

$$\begin{aligned} &\omega_n \subset \omega_{n+1} \quad \text{for all} \quad n \geqslant 1; \\ &\bigcup_{n\geqslant 1} \omega_n \quad \text{is everywhere dense in } I. \end{aligned} \tag{16}$$

Then an application of Theorem 8 yields the convergence property

$$\lim_{n\to\infty} \| \mathrm{SP}_{\omega_n}(f) - f \| = 0$$

in the norm topology of the space $\mathscr{K}^2(I)$ for any $f \in \mathscr{K}^2(I)$.

In the same way, if $(\omega_n)_{n\geq 1}$ is a sequence of meshes on the torus $\mathbf{T}$ with properties analogous to (16), the convergence of the periodic cubic spline interpolation functions

$$\lim_{n\to\infty} \| \mathrm{SPP}_{\omega_n}(f) - f \| = 0$$

follows for any function $f \in \mathscr{K}^2(\mathbf{T})$ with respect to the norm topology of the space $\mathscr{K}^2(\mathbf{T})$.

4. Concluding Remarks

The Examples (i) and (ii) of Section 2 are restricted to the case of cubic splines only for notational convenience. Their extensions to *general polynomial spline functions* are immediate and will not be given here. For the case of L_m-splines, where L_m denotes a linear differential operator of order m with sufficiently smooth real-valued coefficient functions, see the dissertation [5]. Furthermore, it should be observed that the convergence properties proved in Section 3 for sequences of spline approximations are more generally valid for arbitrary *directed families* of spline projectors. This fact will be of importance for spline functions in several variables, i.e., for tensor products of spline systems. The details will be treated in a forthcoming paper.

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Printed in Belgium